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The natural oscillations of an elastic body with a heavy rigid spike-shaped inclusion☆

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ABSTRACT

It is established that oscillations in the low-frequency range are characteristic for a body with a heavy-rigid spike-shaped inclusion, and corresponding modes mainly occur as flexural deformations of the tip of the spike, localized close to its vertex.

1. The oscillations of composite bodies with contrasting properties

Suppose Ω^0 and $\Omega^1=\Omega\backslash\Omega^0$ are plane anisotropic inhomogeneous bodies, for which the outer boundary $\Gamma=\partial\Omega$ is smooth, while the interface of the bodies $\Gamma^0=\partial\Omega^0$ is smooth everywhere, apart from the origin of coordinates O, close to which the region Ω^0 is specified by the inequalities

$$x_1>0,\quad -b_{-}x_1^{1+\gamma}<x_2<b_{+}x_1^{1+\gamma} \tag{1.1}$$

Here $x=(x_1,x_2)$ is a fixed system of Cartesian coordinates (they are made dimensionless by scaling: the characteristic dimension of the body Ω^0 is taken to be unity), and $b=b_{+}+b_{-}$ and γ are positive numbers. In other words, the inclusion Ω^0 has a zero tip angle – a spike, directed outwards (Fig. 1).

The problem of the oscillations of a composite body $\Omega=\Omega^0\cup\Omega^1$ has the following matrix form[1,2]

$$D(-\nabla_x)^{\top}A^i(x)D(\nabla_x)u^i(x)=\Lambda\rho^i(x)u^i(x),\quad x\in\Omega^i,\quad i=0,1 \tag{1.2}$$

$$u^0(x)=u^1(x),\quad D(n(x))^{\top}A^0(x)D(\nabla_x)u^0(x)=D(n(x))^{\top}A^1(x)D(\nabla_x)u^1(x),$$
$$x\in\Gamma^0 \tag{1.3}$$

$$D(n(x))^{\top}A^1(x)D(\nabla_x)u^1(x)=0,\quad x\in\Gamma \tag{1.4}$$

Here $u=(u_1,u_2)^{\top}$ is the column of displacements, T is the sign of transposition, u^i is the constriction of the field u on the body Ω^i, n is the unit vector (column) of the outward normal of the boundaries $\partial\Omega^0$ and $\partial\Omega$, A^i is the matrix of the elastic moduli, symmetrical and positive definite in the set $\bar{\Omega}^i=\Omega^i\cup\partial\Omega^i$, ρ^i is the density of the material, which is a smooth positive function in the set $\bar{\Omega}^i$, and $D(\nabla_x)_u=(\varepsilon_{11},\sqrt{2}\varepsilon_{12},\varepsilon_{22})^{\top}$ is the deformation column, where

$$D(\nabla_x)^{\top}=\begin{pmatrix}\partial_1 & 2^{-1/2}\partial_2 & 0\\ 0 & 2^{-1/2}\partial_1 & \partial_2\end{pmatrix},\quad \partial_j=\frac{\partial}{\partial x_j},\quad \nabla_x=\begin{pmatrix}\partial_1\\ \partial_2\end{pmatrix} \tag{1.5}$$

Since $A^iD(\nabla_x)u^i=(\sigma^i_{11},\sqrt{2}\sigma^i_{12},\sigma^i_{22})^{\top}$ is the stress column, conditions (1.3) and (1.4) denote that along the contour Γ^0 ideal contact of the bodies Ω^0 and Ω^1 is assumed, while the outer surface of the body is load-free.

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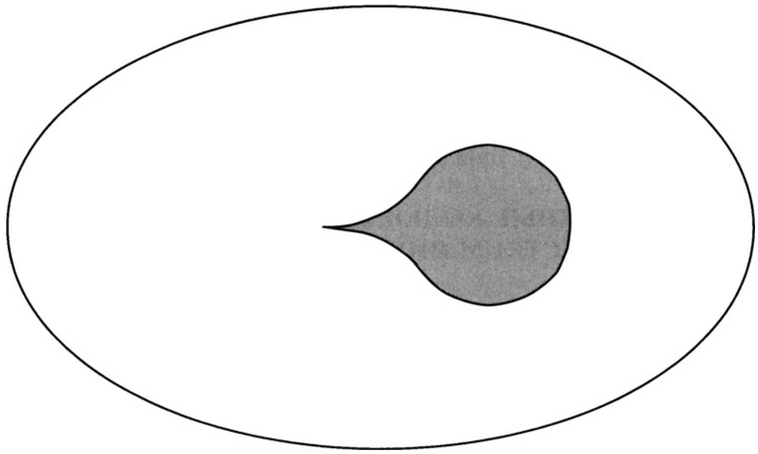

Fig. 1.

It is well known that the eigenvalues Λ_k of problem (1.2)–(1.4), i.e., the squares of the frequencies of the natural oscillations of the composite body, form an infinitely large series

$$0 = \Lambda_1 = \Lambda_2 = \Lambda_3 < \Lambda_4 \le \Lambda_5 \le \ldots \le \Lambda_k \le \ldots \to +\infty \tag{1.6}$$

The corresponding eigenvectors, i.e., the modes of natural oscillations, which belong to the Sobolev space $H^1(\Omega)$, may be subject to the conditions of orthogonality and normalization

$$(\rho u^{(j)}, u^{(k)})_\Omega = \delta_{jk}, \quad j, k = 1, 2, \ldots \tag{1.7}$$

Here $(,)_\Omega$ is a scalar product in Lebejgue space $L_2(\Omega)$. The eigenvalues are indicated in series (1.6) taking their multiplicities into account; in particular, the zero eigenvalue has a three-dimensional natural subspace, consisting of rigid displacements $(c_1 - c_0x_2, c_2 + c_0x_1)^\top$.

We will assume that the parts Ω^1 and Ω^0 of the composite body possess contrasting properties

$$A^1(x) = \tau A^{10}(x), \quad \rho^1(x) = \tau^{1+\beta}\rho^{10}(x) \tag{1.8}$$

Here $\tau > 0$ is a small parameter, $\beta \ge 0$, and A^{10} and ρ^{10} are characteristics, comparable in order of magnitude with the characteristics A^0 and ρ^0 of the body Ω^0. In other words, the material of the inclusion Ω^0 is more rigid and heavier than the material of the frame Ω^1. A large number of publications (Refs 3, 4, etc.) are devoted to investigating the spectra of singularly perturbed problems of this kind. It follows from the results in Refs 3, 5 and 6, in particular, that in the case of a smooth contour Γ^0 of separation of the bodies for the eigenvalues Λ_k^τ of problem (1.2)–(1.4) with condition (1.8), the following convergences occur

$$\Lambda_k^\tau \to \Lambda_k^0, \quad \tau \to +0 \; k = 1, 2, \ldots \tag{1.9}$$

If $\beta > 0$, then Λ_k^0 are terms of the series (1.6) of the eigenvalues of the problem of the oscillations of an isolated body Ω^0 with a free surface

$$D(-\nabla_x)^\top A^0(x) D(\nabla_x) \upsilon^0(x) = \Lambda^0 \rho^0(x) \upsilon^0(x), \quad x \in \Omega^0 \tag{1.10}$$

$$D(n(x))^\top A^0(x) D(\nabla_x) \upsilon^0(x) = 0, \quad x \in \Gamma^0 \tag{1.11}$$

If $\beta = 0$, the sequence (1.6) of the limits Λ_k^0 consists of the eigenvalues of two problems: problem (1.10), (1.11) for the body Ω^0 and the mixed boundary-value problem for the body Ω^1

$$D(-\nabla_x)^\top A^{10}(x) D(\nabla_x) \upsilon^1(x) = \Lambda^0 \rho^{10}(x) \upsilon^1(x), \quad x \in \Omega^1 \tag{1.12}$$

$$D(n(x))^\top A^{10}(x) D(\nabla_x) \upsilon^1(x) = 0, \quad x \in \Gamma \tag{1.13}$$

$$\upsilon^1(x) = 0, \quad x \in \Gamma^1 \tag{1.14}$$

Note that the conditions for ideal contact (1.3) of the contour Γ^0 are split up as $\tau \to +0$: the first of them reduces the boundary condition in displacements (1.14) on the surface of the less rigid body, while the second reduces the boundary condition in the stresses (1.11) on the surface of the less rigid body. Thus, if relations (1.8) when $\beta = 0$ include the last parameter τ, the limit as $\tau \to +\infty$ leads to eigenvalue problems for the isolated bodies Ω^0 and Ω^1 with constrained and free surfaces respectively.

When $\beta > 0$ the eigenvalues Γ of problem (1.12)–(1.14), as before, participate in the formation of the asymptotic structure ($\tau \to +0$) of the spectrum of problem (1.12)–(1.14), namely: according to the well-known results,[5,6] for any such eigenvalue Λ_n^* the eigenvalue $\Lambda_{N(\tau)}^\tau$ is obtained, subject to the limit

$$\left|\Lambda_{N(\tau)}^\tau - \Lambda_n^*\right| \le c_n\sqrt{\tau} \tag{1.15}$$

Nevertheless, the number N(τ) of this number in a sequence of the form (1.6) increases without limit as $\tau \to +0$. In other words, when $\beta > 0$ formula (1.9) serves to describe the asymptotic form in the low-frequency band of the spectrum, while formula (1.15) serves to describe a series of eigenvalues with stable asymptotic forms from the high-frequency band.

It is shown in this paper that the facts, verified in Refs 5 and 6 for smooth separating surfaces, cease to be true for spike-shaped inclusions when $\gamma \geq 1$ (see formula (1.1)). The reasons for the change in the asymptotic behaviour of the eigenvalues of problem (1.2)–(1.4) when $\tau \to +0$ will be explained in the next section, while the remaining sections are devoted to constructing the asymptotic form of the eigenvalues Λ_k^τ with the additional condition

$$\gamma > 1 \tag{1.16}$$

The question of the asymptotic structure of the spectrum when $\gamma = 1$ remains open.

2. Spike-shaped inclusions

If the boundary $\Gamma^0 = \partial\Omega^0$ is Lipschitzian, the Korn inequality

$$\left\|u;\, H^1(\Omega^0)\right\| \leq c\left(\left\|D(\nabla_x)u;\, L_2(\Omega^0)\right\|^2 + \left\|u;\, L_2(\omega^0)\right\|^2\right)^{1/2} \tag{2.1}$$

is satisfied (see Refs 7, 2, etc.). Here we have used the notation (1.5), while $\omega^0 \subset \Omega^0$ is a subregion of positive area. The norm on the right-hand side of formula (2.1) generates an energy class E(Ω^0), since the term $||D(\nabla_x)u;\, L_2(\Omega^0)||^2$ is equivalent to the elastic energy functional. According to classical results of the theory of operators in Hilbert space (see, for example, Ref. 8), the compactness of the imbedding $E(\Omega^0) \subset L_2(\Omega^0)$ ensures that the spectrum of problem (1.2)–(1.4) is discrete as well as the other properties of sequence (1.6) of its eigenvalues.

When there is a spike (1.1) present, the boundary Γ^0 loses the Lipschitzian property, and the Korn inequality (2.1) ceases to be true (see Ref. 2, Section 3.1). To check this conclusion, consider the sequence of fields $u^{(m)}$ ($m = 1,2,\ldots$) with components

$$u_2^{(m)}(x) = m^{\delta + 3\gamma/2} x_1^{\delta} \psi(m x_1), \quad u_1^{(m)}(x) = -m^{\delta + 3\gamma/2} x_2 \partial_1 (x_1^{\delta} \psi(m x_1)) \tag{2.2}$$

Here $\Psi \neq 0$ is a smooth function with a carrier in the section (d/2,d) and d > 0 is small fixed length. Suppose $\omega^0 = \Omega^0 \backslash B_d$, where B_d is a sphere of diameter 2d and centre O.

It is easy to show that

$$\left\|u^{(m)};\, E(\Omega^0)\right\| = O(1), \quad \left\|\nabla_x u^{(m)};\, L_2(\Omega^0)\right\| = O(m^{\gamma}), \quad \left\|u^{(m)};\, L_2(\Omega^0)\right\| = O(m^{\gamma - 1}) \tag{2.3}$$

Consequently, inequality (2.1) with a constant c common for all the fields $u \in H^1(\Omega^0)$ is impossible. Moreover, taking $\psi = 1$ and m = 1 in formula (2.2), it can be seen that the finiteness of the norms in the list (2.3) is ensured respectively by the formulae

$$\delta > -3\gamma/2, \quad \delta > -\gamma/2, \quad \delta > -1 - \gamma/2$$

Hence, the space E(Ω^0) is not contained in the space $H^1(\Omega^0)$ for any $\Gamma > 0$ and is not contained in the space $L_2(\Omega^0)$ when $\gamma > 1$. In view of the weighted Korn inequality,[9] imbedding of $E(\Omega^0) \subset L_2(\Omega^0)$ occurs when $\gamma = 1$, but the first and third relations of (2.3) for the sequence of fields (2.2) demonstrate that it cannot be compact.

These facts indicate (Ref. 8, Section 10.1) the occurrence of a continuous spectrum of problem (1.2)–(1.4) on the real non-negative semiaxis, i.e., the convergences cannot even be given a meaning. We further verified that, when condition (1.16) is satisfied, the following formulae hold

$$\tau^{-2(1 - 1/\gamma)/3} \Lambda_{k+3}^{\tau} \to M_k, \quad \tau \to +0, \quad k = 1, 2, \ldots \tag{2.4}$$

Moreover, an eigenvalue problem is described, the eigenvalues of which act as limits in formulae (2.4) and form the sequence

$$0 < M_1 \leq M_2 \leq \ldots \leq M_k \leq \ldots \to +\infty \tag{2.5}$$

The exponent $-2(1-1/\gamma)/3$ of the power of the small parameter τ is negative, since $\gamma > 1$, and so, for any fixed number k, the eigenvalue Λ_k^τ of problem (1.2)–(1.4) for a composite body with a rigid and heavy spike-shaped inclusion approaches zero as $\tau \to +\infty$. In other words, for small $\tau > 0$ the low-frequency band of the spectrum is compressed to a point $\Lambda = 0$ and contains a large number of closely situated frequencies of natural oscillations.

In the limit problem ($\tau = 0$) a continuous spectrum occurs exclusively in the case of heavy rigid spike-shaped inclusions. Thus, if a large parameter τ occurs in relations (1.8), the spectrum of the limit problem $\tau \to +\infty$, consisting of the system of differential equations (1.10) and the boundary conditions $v^0 = 0$ on Γ^0, remains discrete, thanks to the Dirichlet conditions on the boundary of the regions Ω^0, which enables as to verify the Korn inequalities and the compactness of the required imbeddings without difficulty. Note also that for any $\tau \in (0, +\infty)$ the stresses and strains for natural oscillations of a composite body $\Omega^0 \cup \Omega^1$ remain bounded at the vertex of the spike. Asymptotic expansions of the elastic fields were formerly constructed in its neighbourhood[10,11] and then proved.[12]

3. Asymptotic anzatzes

We will introduce the extended coordinates

$$\xi = (\xi_1, \xi_2) = \tau^{-1/(3\gamma)} x \tag{3.1}$$

Relations (1.1) take the form

$$\eta := \xi_1 > 0, \quad -b_-\xi_1^{1+\gamma} < \tau^{-1/3}\xi_2 < b_+\xi_1^{1+\gamma} \tag{3.2}$$

Hence, the spike-shaped inclusion becomes thin – we take its "thickness" $\tau^{1/3}$ as the new small parameter h and we introduce the superfast variable

$$\zeta = h^{-1}\xi_2 = \tau^{-(1-1/\gamma)/3}x_2 \in \Upsilon(\eta) = (-b_-\eta^{1+\gamma}, b_+\eta^{1+\gamma}) \tag{3.3}$$

On the set Π^h, given by formula (3.2), we take the asymptotic anzatzes for solving eigenvalue problems of the theory of elasticity in thin regions (see Refs 13 and 2, Chapter 7 etc.)

$$\Lambda^\tau = \tau^{\beta-2/(3\gamma)}h^2M + \ldots = \tau^{\beta+2(1-1/\gamma)/3}M + \ldots \tag{3.4}$$

$$u^0(x) = W^0(\eta) + hW^1(\eta,\zeta) + h^2W^2(\eta,\zeta) + h^3W^3(\eta,\zeta) + h^4W^4(\eta,\zeta) + \ldots \tag{3.5}$$

$$W^0(\eta) = e_2w_2(\eta), \quad W^1(\eta,\zeta) = e_1(w_1(\eta) - \zeta\partial_\eta w_2(\eta)) \tag{3.6}$$

Here $e_p = (\delta_{p1},\delta_{p2})^\top$ is the unit vector of the x_p axis, and M and $w = (w_1,w_2)^T$, W^q is the number and vector functions to be determined later. The asymptotic anzatz for the displacement field u^1 in the region Ω^1 is matched with anzatz (3.5) and has the form

$$u^1(x) = \upsilon(\xi) + h\upsilon^1(\xi) + h^2\upsilon^2(\xi) + \ldots \tag{3.7}$$

We substitute expansions (3.5), (3.7) and (3.4) into Eqs (1.2)–(1.4) and we collect coefficients of like powers of the small parameter τ, forming problems for determining the terms of the anzatzes. Because of the change to extended coordinates in formula (3.1), the matrices A^i of the elastic moduli and the density ρ^i should be frozen at the vertex of the spike, and hence, for brevity, we will write A^i and ρ^i instead of $A^i(O)$ and $\rho^i(O)$. The vector functions v and v^q depend on the variables ξ in the plane $\mathbb{R}^2$ with a semi-infinite cut $\Sigma = \{\xi:\xi_1 > 0, \xi_2 = 0\}$, to which the set $\bar{\Pi}_h$ contracts as $h \to +0$. From the first conjugation condition (1.3) and relations (3.5)–(3.7) we derive that

$$u = \upsilon_1(\eta, \pm 0), \quad w_2(\eta) = \upsilon_2(\eta, \pm 0), \quad \eta > 0 \tag{3.8}$$

Moreover, the vector function $\upsilon^0 = (\upsilon_1^0, \upsilon_2^0)^\top$ satisfies the following system of differential equations

$$D(-\nabla_\xi)^\top A^{10}D(\nabla_\xi)\upsilon(\xi) = 0, \quad \xi \in \mathbb{R}^2\backslash\bar{\Sigma} \tag{3.9}$$

Here we have taken into account the fact that the left-hand side of system (3.9) serves as a factor for $\tau^{1-2\gamma/3}$ on the left-hand side of relation (1.2) with i = 1, while the inertial term on the right-hand side of this relation requires a higher order $\tau^{1+\beta+2(1-1/\gamma)3}$ according anzatzes (3.4) and (3.7). The deficient condition of conjugation on the line Σ will be obtained when constructing the terms of anzatz (3.5).

The following relations are satisfied for the unit vectors $n^\pm$ of the normal to the arcs $\Gamma^\pm$, forming the spike (1.1),

$$n^\pm(x) = (\pm\hat{n}_\pm(x_1)^{-1/2}, \hat{n}_\pm(x_1)^{-1/2}b_\pm(1+\gamma)x_1^\gamma), \quad \hat{n}_\pm(x_1) = 1 + b_\pm^2(1+\gamma)^2x_1^{2\gamma}$$

The change to the coordinates η and ζ is accompanied by the following splitting of the matrix differential operators

$$\begin{aligned}
&D(-\nabla_x)^\top A^0D(\nabla_x) = \tau^{-2/(3\gamma)}h^{-2}(L^0(\partial_\zeta) + hL^1(\partial_\eta,\partial_\zeta) + h^2L^2(\partial_\eta))\\
&\hat{n}_\pm(x_1)^{1/2}D(n^\pm(x))^\top A^0D(\nabla_x) =\\
&= \tau^{-1/(3\gamma)}h^{-1}(N^{0\pm}(\partial_\zeta) + hN^{1\pm}(\eta,\partial_\eta,\partial_\zeta) + h^2N^{2\pm}(\eta,\partial_\eta))
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
&L^0(\partial_\zeta) = -A^0_{(22)}\partial_\zeta^2, \quad L^1(\partial_\eta,\partial_\zeta) = -(A^0_{(21)} + A^0_{(12)})\partial_\eta\partial_\zeta, \quad L^2(\partial_\eta) = -A^0_{(11)}\partial_\eta^2\\
&N^{0\pm}(\partial_\zeta) = \pm A^0_{(22)}\partial_\zeta, \quad N^{1\pm}(\eta,\partial_\eta,\partial_\zeta) = \pm A^0_{(21)}\partial_\eta + b_\pm(1+\gamma)\eta^\gamma A^0_{(12)}\partial_\zeta\\
&N^{2\pm}(\eta,\partial_\eta) = b_\pm(1+\gamma)\eta^\gamma A^0_{(11)}\partial_\eta\\
&A^0_{jk} = D(e_j)^\top A^0D(e_k)
\end{aligned} \tag{3.11}$$

Note that, in view of proposition (1.8) and expansion (3.7), the right-hand side of the second conjugation condition (1.3) on the arcs $\Gamma^\pm$ is equal to

$$\tau^{1-1/(3\gamma)}D(\pm e_2)^\top A^{10}D(\nabla_\xi)\upsilon(\xi_1, \pm 0) + \ldots = \tau^{1-1/(3\gamma)}G^{\upsilon_\pm}(\eta) + \ldots \tag{3.12}$$

Thus, substituting the anzatzes (3.5) and (3.4) and the splittings (3.10) into system of differential equations (1.2) with i = 0 and into the conjugation conditions mentioned, we collect the coefficients of like powers of the small parameter τ and arrive at a recurrent sequence of problems on the cut $\gamma(\eta)$ with parameter $\eta > 0$

$$\begin{aligned}
&L^0(\partial_\zeta)W^q(\eta,\zeta) = F^q(\eta,\zeta) := -L^1(\partial_\eta,\partial_\zeta)W^{q-1}(\eta,\zeta) - L^2(\partial_\eta)W^{q-2} +\\
&+ \delta_{q4}M\rho^0W^0(\eta), \quad \zeta \in \Upsilon(\eta)
\end{aligned} \tag{3.13}$$

$$N^{0\pm}(\partial_\zeta)W^q(\eta,\zeta) = G^{q\pm}(\eta) := -N^{1\pm}(\eta,\partial_\eta,\partial_\zeta)W^{q-1}(\eta,\zeta) - N^{2\pm}(\eta,\partial_\eta)W^{q-2}(\eta,\zeta) + \\ +\delta_{q4}G^{\pm}(\eta), \quad \zeta = \pm b_\pm\eta^{1+\gamma} \tag{3.14}$$

Here $q = 0, \ldots, 4$ and $W^j = 0$ when $j < 0$. Note that the coefficients in formulae (3.1) and (3.4) were chosen so that the principal parts of the inertial terms and the discrepancies (3.12), left by expansion (3.7) in the region Ω^1, fall on the right-hand side of the problem for determining the term W^4 of expansion (3.5) in the region Ω^0, namely, at this step of the algorithm for constructing the asymptotic form, correctly formulated resultant eigenvalue problems in the theory of plates and columns arise (see Ref. 2, Chapter 7).

It is obvious that the vector functions (3.6) satisfy problems (3.14) when $q = 0$ and $q = 1$. When $q = 2$, in view of relations (3.12) and the key equality

$$D(e_2)e_1 = D(e_1)e_2 \tag{3.15}$$

ensured by the structure of the matrix D from formula (1.5), we have

$$F^2(\eta,\zeta) = -A^0_{(21)}e_1\partial^2_\eta w_2(\eta), \quad G^{2\pm}(\eta) = \mp A^0_{(21)}e_1(\partial_\eta w_1(\eta) - \zeta\partial^2_\eta w_2(\eta))$$

Consequently, the solution is given by the formula

$$W^2(\eta,\zeta) = -(A^0_{(22)})^{-1}A^0_{(21)}e_1\left(\zeta\partial_\eta w_1(\eta) - \frac{1}{2}\zeta^2\partial^2_\eta w_2(\eta)\right) \tag{3.16}$$

In problem (3.13) and (3.14) when $q = 3$ the right-hand sides are

$$F^3 = (A^0_{(21)} + A^0_{(12)})\partial_\eta\partial_\zeta W^2 + A^0_{(11)}\partial^2_\eta W^1 \\ G^{3\pm} = \mp A^0_{(21)}\partial_\eta W^2 + b_\pm(1+\gamma)\eta^\gamma(A^0_{(12)}\partial_\zeta W^2 + A^0_{(11)}\partial_\eta W^1) \tag{3.17}$$

We convert the condition for this problem to be solvable

$$0 = \int\limits_{\Upsilon(\eta)} F^3(\eta,\zeta)d\zeta + G^{3+}(\eta) + G^{3-}(\eta) = \\ = \frac{\partial}{\partial\eta}\int\limits_{\Upsilon(\eta)}(A^0_{(12)}\partial_\zeta W^2(\eta,\zeta) + A^0_{(11)}\partial_\eta W^1(\eta,\zeta))d\zeta \tag{3.18}$$

According to the key equality (3.15), the last formula of (3.11) and relations (3.6) and (3.16), the second component of the integrand in the integral over the cut $\gamma(\eta)$ vanishes:

$$e_2^\top(A^0_{(12)}\partial_\zeta W^2 + A^0_{(11)}\partial_\eta W^1) = (-e_1^\top D(e_2)^\top A^0 D(e_2)(A^0_{(22)})^{-1}D(e_2)^\top A^0 D(e_1) + \\ + e_1^\top D(e_2)^\top A^0 D(e_1))e_1(\partial_\eta w_1(\eta) - \zeta\partial^2_\eta w_2(\eta)) = 0$$

Carrying out the integration of the first component and using all the same formulae, with the exception of equality (3.15), we obtain the following ordinary differential equation

$$-\frac{\partial}{\partial\eta}ab\eta^{1+\gamma}\frac{\partial w_1}{\partial\eta}(\eta) + \frac{1}{2}\frac{\partial}{\partial\eta}a(b_+^2 - b_-^2)\eta^{2(1+\gamma)}\frac{\partial^2 w_2}{\partial\eta^2}(\eta) = 0, \quad \eta > 0 \tag{3.19}$$

$$a = \hat{e}_1^\top(A^0 - A^0 D(e_2)(A^0_{(22)})^{-1}D(e_2)^\top A^0)\hat{e}_1 \tag{3.20}$$

Since the columns $\hat{e}_1 = (1,0,0)^\top$ and $\hat{c} = D(e_2)(A^0_{(22)})^{-1}D(e_2)^\top A^0\hat{e}_1 = \mathbb{R}^3$ are mutually orthogonal, by virtue of definition (1.5) of the matrix D, the positiveness of the scalar (3.20) follows from the relations

$$\hat{e}_1 + \hat{c} \neq 0, \quad 0 < (\hat{e}_1 + \hat{c})^\top A^0(\hat{e}_1 + \hat{c}) = a$$

For an isotropic material with shear modulus μ and Poisson's ratio $\mu < ½$, the value of (3.20) is equal to $a = 4\mu(1-2v)^{-1}$.

Equation (3.19) shows that

$$\frac{\partial w_1}{\partial\eta}(\eta) = \frac{1}{2}(b_+ - b_-)\eta^{1+\gamma}\frac{\partial^2 w_2}{\partial\eta}(\eta) + c\eta^{-1-\gamma} \tag{3.21}$$

Here the constant c must vanish – otherwise the functions w_1 or w_2 will have impermissible singularities $O(\eta^{-\gamma})$ or $O(\eta^{1-\gamma})$ at the origin of coordinates.

We now put q = 4 and develop one of the conditions for problem (3.13), (3.14) with right-hand sides

$$F^4(\eta,\zeta) = (A^0_{(21)} + A^0_{(12)})\partial_\eta\partial_\zeta W^3 + A^0_{(11)}\partial^2_\eta W^2 + M\rho^0 W^0$$

$$G^{3\pm}(\eta) = \mp A^0_{(21)}\partial_\eta W^3 + b_\pm(1+\gamma)\eta^\gamma(A^0_{(12)}\partial_\zeta W^3 + A^0_{(11)}\partial_\eta W^1) + G^{\upsilon\pm}(\eta)$$

to be solvable. We similarly obtain, by conversions (3.18), that the equality

$$\int_{\Upsilon(\eta)} F^4_2(\eta,\zeta)d\zeta + G^{4+}_2(\eta) + G^{4-}_2(\eta) = 0$$

is equivalent to the relation

$$-M\rho^0 b\eta^{1+\gamma}w_2(\eta) - G^{\upsilon+}_2(\eta) - G^{\upsilon-}_2(\eta) =$$

$$= \frac{\partial}{\partial\eta}\int_{\Upsilon(\eta)} e_2^\top(A^0_{(12)}\partial_\zeta W^3(\eta,\zeta) + A^0_{(11)}\partial_\eta W^2(\eta,\zeta))d\zeta =: S(\eta)$$

We continue the conversions using formulae (3.6), (3.11), (3.15)–(3.17) and (3.13), (3.14) with q = 3. We have

$$S(\eta) = \frac{\partial}{\partial\eta}\int_{\Upsilon(\eta)} e_1^\top(A^0_{(22)}\partial_\zeta W^3(\eta,\zeta) + A^0_{(21)}\partial_\eta W^2(\eta,\zeta))d\zeta =$$

$$= -\frac{\partial}{\partial\eta}\Bigg(\int_{\Upsilon(\eta)} \zeta e_1^\top(A^0_{(22)}\partial^2_\zeta W^3(\eta,\zeta) + A^0_{(21)}\partial_\zeta\partial_\eta W^2(\eta,\zeta))d\zeta -$$

$$-\zeta e_1^\top(A^0_{(22)}\partial_\zeta W^3(\eta,\zeta) + A^0_{(21)}\partial_\eta W^2(\eta,\zeta))\Big|^{b+\eta^{1+\gamma}}_{\zeta=-b-\eta^{1+\gamma}}\Bigg) =$$

$$= \frac{\partial}{\partial\eta}\Bigg(\int_{\Upsilon(\eta)} \zeta e_1^\top(A^0_{(12)}\partial_\eta\partial_\zeta W^2(\eta,\zeta) + A^0_{(11)}\partial^2_\eta W^1(\eta,\zeta))d\zeta -$$

$$-b_\pm(1+\gamma)\eta^\gamma e_1^\top(A^0_{(12)}\partial_\zeta W^2(\eta,\zeta) + A^0_{(11)}\partial_\eta W^1(\eta,\zeta))\Big|^{b+\eta^{1+\gamma}}_{\zeta=-b-\eta^{1+\gamma}}\Bigg) =$$

$$= \frac{\partial^2}{\partial\eta^2}\int_{\Upsilon(\eta)} \zeta e_1^\top(A^0_{(12)}\partial_\zeta W^2(\eta,\zeta) + A^0_{(11)}\partial_\eta W^1(\eta,\zeta))d\zeta =$$

$$= -\frac{\partial^2}{\partial\eta^2}a\int_{\Upsilon(\eta)}\left(\zeta^2\frac{\partial^2 w_2}{\partial\eta^2}(\eta) - \zeta\frac{\partial w_1}{\partial\eta}(\eta)\right)d\zeta =$$

$$= -\frac{a}{3}(b_+^3 + b_-^3)\frac{\partial^2}{\partial\eta^2}\eta^{3(1+\gamma)}\frac{\partial^2 w_2}{\partial\eta^2}(\eta) + \frac{a}{2}(b_+^2 - b_-^2)\frac{\partial^2}{\partial\eta^2}\eta^{2(1+\gamma)}\frac{\partial w_1}{\partial\eta}(\eta) \tag{3.22}$$

The calculations are completed. Collecting the relations (3.8), (3.12) and (3.21), (3.22), we form the resulting problem: the system of differential equations (3.9) is provided with boundary conditions and matching conditions on the semi-infinite section Σ

$$\upsilon_1(\zeta_1,\pm 0) = 0, \quad \upsilon_2(\xi_1,+0) = \upsilon_2(\xi_1,-0), \quad \xi_1 > 0 \tag{3.23}$$

$$e_2^\top D(-e_2)^\top A^{10}(D(\nabla_\xi)\upsilon(\xi_1,+0) - D(\nabla_\xi)\upsilon(\xi_1,-0)) =$$

$$= M\rho^0 b\xi_1^{1+\gamma}\upsilon_2(\xi_1,0) - \frac{ab^3}{12}\frac{\partial^2}{\partial\xi_1^2}\xi_1^{3(1+\gamma)}\frac{\partial^2\upsilon_2}{\partial\xi_1^2}(\xi_1,0), \quad \xi_1 > 0 \tag{3.24}$$

4. The spectrum of the resulting problem

Suppose $\mathbb{C}$ is the lineal of vector functions, infinitely differentiable up to the boundary in the region $\mathbb{R}^2\backslash\bar{\Sigma}$, which possess compact carriers, vanish in the region of the origin of coordinates and satisfy stable conditions (3.23). The integral identity, which ensures a variational formulation of the eigenvalue problem (3.9), (3.23), (3.24), is obtained by multiplying system (3.9) by the vector function $V \in \mathbb{C}$ and subsequent integration by parts in the region $\mathbb{R}^2 \setminus \bar{\Sigma}$ and over the ray Σ, taking into account the conjugation condition (3.24). It has the form

$$(A^{10}D(\nabla_\xi)\upsilon, D(\nabla_\xi)V)_{\mathbb{R}^2} + a\frac{b^3}{12}\left(\xi_1^{3(1+\gamma)}\frac{\partial^2\upsilon_2}{\partial\xi_1^2}, \frac{\partial^2 V_2}{\partial\xi_1^2}\right)_\Sigma = M\rho^0 b(\xi_1^{1+\gamma}\upsilon_2, V_2)_\Sigma \tag{4.1}$$

We will denote by **H** the supplement of the lineal **C** with respect to the norm, generated by the quadratic form from the left-hand side of identity (4.1), while **L** is the supplement of the lineal $C_c^\infty(0, +\infty)$ of the smooth functions with compact carriers on the ray $(0, +\infty)$ with respect to the weighted norm $\|\xi_2^{(1+\gamma)/2}v, L_2(0, +\infty)\|$, produced by the form of the right-hand side of the identity. If $v \in \mathbb{C}$, the use of the Korn inequality in a plane, verified by a Fourier transformation, and the known one-dimensional Hardy inequalities on the ray, leads to the relations

$$\left\|\nabla_\xi \upsilon; L_2(\mathbb{R}^2 \setminus \Sigma)\right\|^2 \le c(A^{10}D(\nabla_\xi)\upsilon, D(\nabla_\xi)\upsilon)_{\mathbb{R}^2} \tag{4.2}$$

$$\left\|\xi_1^{(3\gamma-1)/2}\upsilon_2; L_2(\Sigma)\right\| \le c\left\|\xi_1^{(3\gamma+1)/2}\frac{\partial\upsilon_2}{\partial\xi_1}; L_2(\Sigma)\right\| \le C\left\|\xi_1^{3(\gamma+1)/2}\frac{\partial^2\upsilon_2}{\partial\xi_1^2}; L_2(\Sigma)\right\| \tag{4.3}$$

Moreover, versions of the Friedrichs inequality, one-dimensional on the arc $(0, 2\pi) \ni \varphi$ (the angular variable) and two-dimensional in a circle, and also the boundary condition for the components v_1 and relation (4.3) for the components v_2 give the limit

$$\left\|(1 + |\xi|)^{-1}\upsilon; L_2(\mathbb{R}^2)\right\| \le c(\left\|\nabla_\xi\upsilon; L_2(\mathbb{R}^2)\right\| + \left\|\xi_1^{(3\gamma-1)/2}\upsilon_2; L_2(\Sigma)\right\|) \tag{4.4}$$

Finally, formulae (4.2) and (4.4) enable us to process the following norm on a finite part of the arc Σ

$$\left\|(1 + |\xi|)^{-1/2}\upsilon_2; L_2(\Sigma)\right\| \le c\|\upsilon; \mathbf{H}\| \tag{4.5}$$

The weighting factors in the norms of the left-hand sides of formulae (4.3) and (4.5) predominate one over the other in the intervals $(1, +\infty)$ and $(0,1)$ respectively. Here $3\gamma - 1 > \gamma + 1$, in view of requirement (1.16), and so the imbedding $\mathbf{H} \subset \mathbf{L}$ is compact.

Inequalities (4.2)–(4.5) enable us to determine the equivalent weighted norm in the Hilbert space **H** and to transfer by closure to the test functions $V \in \mathbf{H}$ in integral identify (4.1).

Suppose $(,)_\mathbf{H}$ is the scalar product in **H** space. We will define the operator $\mathbf{K}$: $\mathrm{H} \to \mathbf{H}$ by the formula

$$(\mathbf{K}\upsilon; V)_\mathbf{H} = \rho^0 b(\xi_1^{1+\gamma}\upsilon_2; V_2)_\Sigma, \quad \upsilon, V \in \mathbf{H}$$

According to the facts presented above, **K** is a compact operator, and is also non-negative, symmetrical and continuous, and so it is self-conjugate. Its kernel is identical with the infinite dimensional subspace $\mathbf{H}_0$ of the vector fields $v \in \boldsymbol{H}$, which vanish on the ray Σ. We will introduce the new spectral parameter $\mu = \mathrm{M}^{-1}$ and, taking the above notation into account, we will rewrite the integral identity (4.1) as an abstract equation in the Hilbert space H

$$\upsilon = \mu\mathbf{K}\upsilon \tag{4.6}$$

By virtue of the properties of the operator **K** and well-known results of the theory of operators in Hilbert space (see, for example, Ref. 8), Eq. (4.6) has a point $\mu = 0$ of the existing spectrum and an infinitely decreasing sequence of positive eigenvalues

$$\mu_1 \ge \mu_2 \ge \ldots \ge \mu_k \ge \ldots \to +0 \tag{4.7}$$

The infinitely increasing sequence (2.5) of eigenvalues $M_k = \mu_k^{-1}$ corresponds to problem (4.1) or, which is the same, problem (3.9), (3.23), (3.24). Note that the eigenvalue $\mu = 0$ of Eq. (4.6) has no effect on the spectrum of problem (4.1), since the relations

$$\upsilon \in \mathbf{H}, \quad (A^{10}D(\nabla_\xi)\upsilon, D(\nabla_\xi)V)_{\mathbb{R}^2} = 0 \ (V \in \mathbf{H})$$

imply the equality $v = 0$ for known properties of the solvability of the problem of the theory of elasticity on a plane with friction with rigidly clamped sides (see, for example, Ref. 14).

The eigen vector functions $v^{(1)}, v^{(2)}, \ldots, v^{(k)}, \ldots$ of problem (4.1), which correspond to the eigenvalues (4.7), indicated when taking their multiplicity into account, also belong to the space **H**, and they may be subject to the conditions of orthogonality and normalization

$$\rho^0 b(\xi_1^{1+\gamma}\upsilon_2^{(j)}, \upsilon_2^{(k)})_\Sigma = \delta_{jk}, \quad j, k = 1, 2, \ldots \tag{4.8}$$

Since the constant vector does not belong to the space **H** in view of inequality (4.4), the number $\mathrm{M} = 0$ is not an eigenvalue for problem (4.1), although the pair $v = (0, 1)^\top$ and $\mathrm{M} = 0$ formally make both sides of the integral identity vanish.

5. Notes on the justification of the asymptotic

Procedures have been developed (Ref. 2, Chapter 7) for direct and inverse reductions, which enable estimates to be established of the asymptotic residues with constants, that are independent of the number k of the eigenvalue Λ_k^τ of problem (1.2)–(1.4), i.e., which enable us to indicate the explicit dependence of the majorant in estimates on attributes of the limit spectrum (2.5). Due to its length, in this paper we do not give either the form of these procedures or even a formulation of the results, which require a long list of notation. We will confine ourselves to mentioning only two simple facts and commentaries on them.

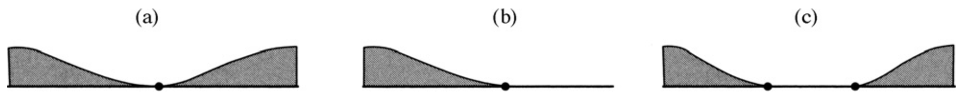

Fig. 2.

First, the asymptotic approximation to the eigenvector function $u^{(k+3)}$ of problem (1.2)–(1.4) is defined by the equations

$$U^{(k+3)0}(x) = \chi(x)\tau^{-(2+\gamma)/(6\gamma)}(W^{(k)0}(\zeta) + hW^{(k)1}(\eta,\zeta) + h^2W^{(k)2}(\eta,\zeta))$$
$$U^{(k+3)1}(x) = \chi(x)\tau^{-(2+\gamma)/(6\gamma)}\upsilon^{(k)}(\xi) \tag{5.1}$$

Here $W^{(k)p}$ are the vector functions of (3.6) and (3.16), constructed with respect to the component $\omega_2^k(\eta) = \upsilon_2^{(k)}(\eta, 0)$ of the eigenvector $v^{(k)}$ of problem (4.1), ξ, η and ζ are fast variables of (3.1)–(3.3), while χ is a cutoff function with a small carrier, equal to unity in the region of the point O. The normalizing factor $\tau^{-(2+\gamma)/(6\gamma)}$ is introduced into definition (5.1) to satisfy the principal condition (1.7); in fact, taking relations (1.1), (1.8), (3.1) and (4.8) into account we obtain

$$(\rho U^{j+3}, U^{(k+3)})_\Omega = \tau^{-(2+\gamma)/(6\gamma)}\left(\rho^0 b\int_0^\infty x_1^{1+\gamma}\upsilon_2^{(j)}(\xi_1, 0)\upsilon_2^{(k)}(\xi_1, 0)dx_1 + \ldots\right) =$$
$$= \rho^0 b(\xi_1^{1+\gamma}\upsilon_2^{(j)}, \upsilon_2^{(k)})_\Sigma + \ldots = \delta_{jk} + \ldots$$

Second, for positive eigenvalues Λ_{k+3}^τ of problem (1.2)–(1.4), relation (2.4), connecting the spectral frequencies (1.6) and (2.5), is satisfied, and also shows that the sharp spike defined by inequalities (1.1) and (1.16) on the contact line of the elastic materials with contrasting properties (1.8) leads to a shift in the spectrum of a composite body in the ultralow frequency band (the quantities Λ_{k+3}^τ are infinitesimal when $\tau \to +0$). Moreover, according to formula (5.1), localization of the modes $u^{(k+3)}$ of the natural oscillations occurs: relations (4.3) and (4.5) establish attenuation of the fields $\upsilon^{(k)}(\xi)$ as $|\xi| \to \infty$. Here, by virtue of formulae (3.6) and (5.1), the natural oscillations at ultralow frequencies mainly occur as a transverse oscillation of the tip of the spike, which excites oscillations of the surrounding material, decaying at a rate $O(|\xi|^{-1/2})$ with distance from the point O (see Ref. 15 in connection with the rate of attenuation).

6. Discussion and open questions

Localization of the modes of natural oscillations in the region of the tip of the broken-off spike agrees completely with the absence of an eigenvalue M = 0 in problem (4.1): the resultant problem (3.9), (3.25), (3.26) is defined as a whole by the structure of the composite body in the region of the point O and is not changed if part of the external boundary Γ is rigidly clamped. In this case the number $\Lambda = 0$ ceases to be an eigenvalue and all the terms of sequence (1.6) become positive, while formulae (2.4) take the form

$$\tau^{-2(1-1/\gamma)/3}\Lambda_j^\tau \to M_j, \quad \tau \to +0, \quad j = 1, 2, \ldots \tag{6.1}$$

The asymptotic analysis and formulation of the results can easily be adapted to other geometrical shapes. Suppose, for example, that the body Ω^1 has a hard and rigid coating Ω^0 (see relations (1.8)), which is worn, i.e., it is thinned out at the point O (Fig. 2*a*). If, at the region of this point, the line Γ^0 coincides with the section of the straight line $\{x:|x_1| < l, x_2 = 0\}$, then, in the neighbourhood of the origin of coordinates, the coating Ω^0 is defined by the formulae

$$|x_1| < l, \quad 0 < x_2 < H(x_1), \quad H(x_1) = |x_1|^{1+\gamma}(b + O(|x_1|)), \quad b > 0$$

The resultant eigen-value problem, similar to problem (3.9), (3.23) and (3.24), then takes the form

$$D(-\nabla_\xi)^\top A^{10}D(\nabla_\xi)\upsilon(\xi) = 0, \quad \xi_1 \in \mathbb{R}, \quad \xi_2 < 0 \tag{6.2}$$

$$e_2^\top D(e_2)^\top A^{10}D(\nabla_\xi)\upsilon(\xi_1, 0) = M\rho^0 b|\xi_1|^{1+\gamma}\upsilon_2(\xi_1, 0) - \frac{ab^3}{12}\frac{\partial^2}{\partial\xi_1^2}|\xi_1|^{3(1+\gamma)}\frac{\partial^2}{\partial\xi_1^2}\upsilon_2(\xi_1, 0)$$
$$\upsilon_1(\xi_1, 0) = 0, \quad \xi_1 \in \mathbb{R} \tag{6.3}$$

If the coating is situated only to the left of the point O (Fig. 2*b*, in which we show the coating with a worn edge), the system of differential equations (6.2) is provided with conditions (6.3) on the semiaxis $\{\xi:\xi_1 > 0, \xi_2 = 0\}$, and when $\xi_2 < 0$ the usual boundary conditions in stresses occur

$$D(e_2)^\top A^{10}D(\nabla_\xi)\upsilon(\xi_1, 0) = 0, \quad \xi_1 < 0 \tag{6.4}$$

When $\gamma > 1$ both eigenvalue problems (6.2), (6.3) and (6.2), (6.4) possess a discrete spectrum of the form (2.5), and formulae (2.4) or (6.1) hold depending on the type of boundary condition on the remaining part of the boundary. In the case of a coating with a gap, shown in Fig. 2*c*, the sequence (2.5) is formed by a combination of the spectra of the two resultant problems, corresponding to two tips.

Localization of the modes of natural oscillations facilitates fracture and provokes peeling of the spike-shaped inclusion Ω^0 from the body Ω^1. If a crack is formed along one of the arcs $\Gamma^\pm$, to be specific Γ^- (compare Figs. 1 and Fig. 3), the asymptotic structure of the spectrum

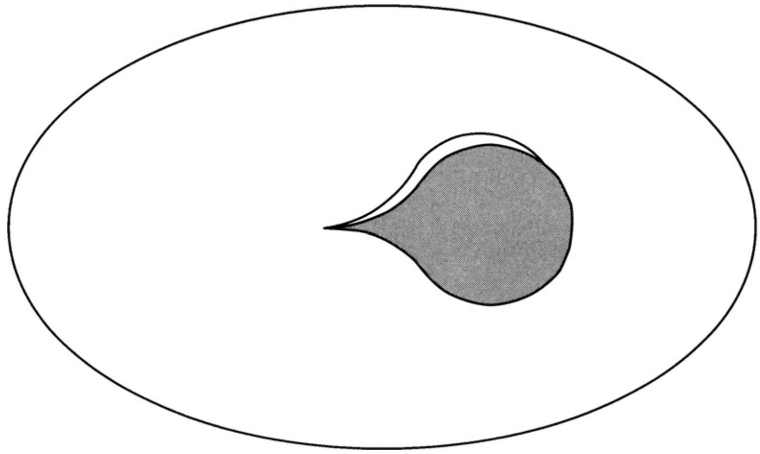

Fig. 3.

(1.6) is only slightly changed: the eigenvalues of the resulting problem, consisting of the system of differential equations (3.9), boundary conditions (6.3) on the upper face Σ^+ and conditions (6.4) on the lower face Σ^- of the semi-infinite cut Σ, become the limits M_k in formulae (2.4). As mentioned in Section 2, the occurrence of cracks along both arcs $\Gamma^\pm$ and the complete disengagement of the spike (1.1) deprives the spectrum of discreteness (see Ref. 8, Section 10.2).

When $\gamma = 1$ both limit problems (1.10), (1.11) and (3.9), (3.23) have a continuous spectrum on the semiaxis $[0, +\infty)$, and it is not known how the eigenvalues Λ_k^τ, $k > 3$ behave as $\tau \to +0$. The question of the asymptotic structure of the spectrum (1.6) of the natural oscillations of a three-dimensional body with a heavy rigid spike-shaped inclusion (Fig. 4)

$$\{x = (x_1, x_2, x_3) : x_3 > 0,\ (x_3^{-1-\gamma} x_1, x_3^{-1-\gamma} x_2) \in \omega \subset \mathbb{R}^2\}$$

remains completely open.

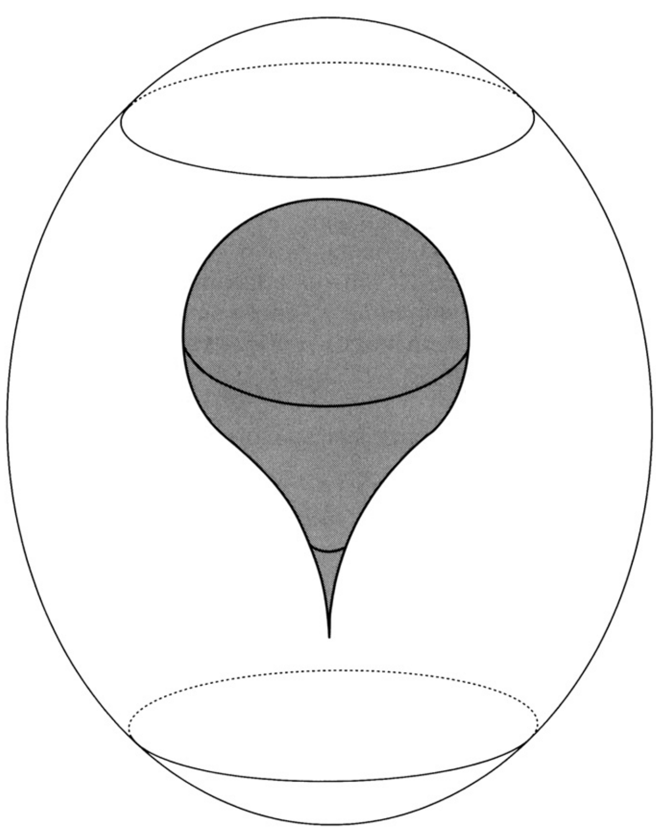

Fig. 4.

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